Proceedings of the Jangjeon Mathematical Society
20 (2017), No. 4. pp. 623 – 639
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2017.20.4.623

JANOWSKI STARLIKENESS AND CONVEXITY

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Abstract. Certain necessary and sufficient conditions are determined for the functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, defined on the open unit disk, to belong to various subclasses of starlike and convex functions. Also discussed are certain sufficient conditions for the normalised analytic functions f of the form $(z/f(z))^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n$, $\mu \in \mathbb{C}$ to belong to the class of Janowski starlike functions.

2010 Mathematics Subject Classification. 30C45, 30C55, 30C80.

Keywords and phrases. starlike functions, convex functions,coefficient conditions, subordination, Janowski starlikeness, Janowski convexity.

1. Introduction and Main Results

This paper deals mainly with the univalent functions having negative coefficients. Precisely, we consider the class $\mathcal{T}$ of analytic univalent functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1}$$

These functions are indeed from the class $\mathcal{A}$ of all normalized functions analytic in $\mathbb{D}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and the class $\mathcal{S}$ of univalent functions in $\mathcal{A}$. For $-1 \leq B < A \leq 1$, let $\mathcal{S}^*[A, B]$ and $\mathcal{C}[A, B]$ be the subclasses of $\mathcal{S}$ consisting of Janowski starlike and Janowski convex functions respectively, defined analytically as:

$$\mathcal{S}^*[A, B] := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

and

$$\mathcal{C}[A, B] := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

When $A = 1 - 2\alpha$, $(0 \leq \alpha < 1)$ and $B = -1$, the above mentioned classes reduce to the classes of starlike functions of order α denoted by $\mathcal{S}^*(\alpha)$ and convex functions of order α denoted by $\mathcal{C}(\alpha)$ respectively. When $A = 0$ and $B = 0$, then $\mathcal{S}^*[0, 0] =: \mathcal{S}^*$ and $\mathcal{C}[0, 0] =: \mathcal{C}$ are the familiar classes of starlike and convex functions. A function $f \in \mathcal{S}$ is *k-uniformly convex* $(k \geq 0)$, if f maps every circular arc γ contained in $\mathbb{D}$ with center ζ, $|\zeta| \leq k$, onto a convex arc. This class of such functions introduced by Kanas and Wisniowska [6] is an extension of the class of uniformly convex functions introduced by Goodman [5]. They showed that f is k-uniformly convex [6, Theorem 2.2, p. 329] (see also [3] for details) if and only if f satisfies the inequality

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right).$$

It is well-known that a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $\sum_{n=2}^{\infty} n|a_n| \leq 1$ is necessarily univalent. This follows easily from the fact that derivative of such functions has positive real part. There are other coefficient conditions that are relevant. Theorem 1.1 extends [5, Theorem 6] to k-uniformly convex functions.

Theorem 1.1 ([6, Theorem 3.3, p. 334]). *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ satisfies the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq 1/(k+2)$ $(k \geq 0)$, then f is k-uniformly convex. The bound $1/(k+2)$ cannot be replaced by a larger number.*

A function $f \in \mathcal{A}$ is *parabolic starlike of order* α if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\alpha + \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

A sufficient coefficient inequality condition for functions to be parabolic starlike is given in the following result.

Theorem 1.2 ([1, Theorem 3.1, p. 564]). *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ satisfies the inequality $\sum_{n=2}^{\infty}(n-1)|a_n| \leq (1-\alpha)/(2-\alpha)$, then f is parabolic starlike of order α. The bound $(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.*

Ali *et al.* investigated the condition on β so that the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$ implies either f is starlike or convex of some positive order. Our primary interest is the investigation of some similar sufficient coefficient conditions for functions to be in the classes $\mathcal{TS}^*[A,B] := \mathcal{T} \cap \mathcal{S}^*[A,B]$, and $\mathcal{TC}[A,B] := \mathcal{T} \cap \mathcal{C}[A,B]$. We obtain here certain necessary and sufficient conditions in terms of the coefficients a_n for the functions in the class $\mathcal{T}$ to be in the classes $\mathcal{TS}^*[A,B]$, $\mathcal{TC}[A,B]$ and $\mathcal{TR}(A,B,\alpha)$. Finally, the reverse implications are investigated for functions to be in the above mentioned subclasses.

First, we obtain some conditions over the coefficients of the function $f \in \mathcal{T}$ to belong the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$.

Theorem 1.3. *Let $-1 \leq B < A \leq 1$ and $f \in \mathcal{T}$ be of the form* (1).

(a) If the function f satisfies any one of the inequalities:

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2(A-B)/(1+A-2B)$;
(2) $\sum_{n=2}^{\infty} (n-1)a_n \leq (A-B)/(1+A-2B)$;
(3) $\sum_{n=2}^{\infty} n^2 a_n \leq 4(A-B)/(1+A-2B)$.
(4) $\sum_{n=2}^{\infty} n a_n \leq 2(A-B)/(2-3B+A)$;

then $f \in \mathcal{TS}^*[A,B]$.

(b) If the function $f \in \mathcal{T}$ satisfies any of the following inequalities:

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(1+A-2B)$;
(2) $\sum_{n=2}^{\infty} n^2 a_n \leq (A-B)/(2-3B+A)$,

then $f \in \mathcal{TC}[A,B]$.

The bounds are sharp.

The previous theorem gave sufficient coefficient conditions for functions to be in $\mathcal{TS}^*[A,B]$ or $\mathcal{TC}[A,B]$. It would be interesting to find the necessary coefficient conditions when the functions belong to these classes. Our next theorem gives some necessary conditions for the functions in $\mathcal{TC}[A,B]$.

Theorem 1.4. *If the function $f \in \mathcal{TC}[A,B]$, then:*

(1) *The inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(1-B)$ holds.*
(2) *The inequality $\sum_{n=2}^{\infty} n^2 a_n \leq 2(A-B)/(1+A-2B)$ holds and the bound is sharp.*

As a consequence of the above theorem and the inequality $2n \leq n^2$ for $n \geq 2$, it can be seen that the inequality $\sum_{n=2}^{\infty} n a_n \leq (A-B)/(1+A-2B)$ holds for the function $f \in \mathcal{TC}[A,B]$. Also, using the inequality $4(n-1) \leq n^2$ for $n \geq 2$, we see that the inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq (A-B)/2(1+A-2B)$ holds and both the bounds obtained here are sharp.

Next, we investigate the class $\mathcal{R}(A,B,\alpha)$ $(\alpha \in \mathbb{R})$ defined by

$$\mathcal{R}(A,B,\alpha) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)}\Big(\alpha \frac{zf''(z)}{f'(z)} + 1\Big) \prec \frac{1+Az}{1+Bz} \right\}. \tag{2}$$

We let $\mathcal{TR}(A,B,\alpha) := \mathcal{T} \cap \mathcal{R}(A,B,\alpha)$. The class $\mathcal{R}(\beta,\alpha) = \mathcal{R}(1-2\beta,-1,\alpha)$ was studied earlier by [2, 8]. Note that $\mathcal{R}(A,B,0) = \mathcal{S}^*[A,B]$. Our next theorem gives a sufficient condition for the functions to belong to the classes $\mathcal{TR}(A,B,\alpha) \cap \mathcal{TS}^*[C,D]$ or $\mathcal{TR}(A,B,\alpha) \cap \mathcal{TC}[C,D]$ respectively.

Theorem 1.5. *Let $\alpha > 0$. If $f \in \mathcal{T}$ satisfies the following inequality:*

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1)a_n \leq (A-B),$$

then the following results hold:

(1) *The function f is in the class $\mathcal{TS}^*[C,D]$ for*

$$C \geq \frac{A-B+D(1-A)+2\alpha D(1-B)}{(1-B)(1+2\alpha)}.$$

The bound obtained is sharp.

(2) *The function f is in the class $\mathcal{TC}[C,D]$ for*

$$C \geq \frac{A-B+D(\alpha-A)+BD(1-\alpha)}{\alpha(1-B)}.$$

The next theorem provides a sufficient coefficient inequality for the functions of the form (1) to belong to the class $\mathcal{TR}(A,B,\alpha)$.

Theorem 1.6. *Let $\alpha \in \mathbb{R}$. If the function f defined by* (1) *satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \frac{2(A-B)}{(A-B)+(1+2\alpha)(1-B)}, \tag{3}$$

then $f \in \mathcal{TR}(A,B,\alpha)$. The bound is sharp.

In our next result, we determine the condition on C so that $\mathcal{TC}[C,D] \subseteq \mathcal{TR}(A,B,\alpha)$.

Theorem 1.7. *Let $\alpha > 0$. If the condition*

$$C \leq \frac{2(A-B)+(1+2\alpha-3A+2B-2\alpha B)D}{(1-A+2\alpha(1-B))}$$

holds, then $\mathcal{TC}[C,D] \subseteq \mathcal{TR}(A,B,\alpha)$.

Finally, the following are the necessary conditions for the functions to belong to the class $\mathcal{TR}(A,B,\alpha)$.

Theorem 1.8. *Let $-1 \leq B < A \leq 1$, and $\alpha \in \mathbb{R}$. If the function $f \in \mathcal{TR}(A,B,\alpha)$, then*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(\alpha(1-B))$, *where* $\alpha > 0$

(2) $\sum_{n=2}^{\infty}(n-1)a_n \leq \gamma$ *where*

$$\gamma = \begin{cases} \frac{A-B}{(1-B)(1-\alpha)}, & (1+3\alpha)B < 3\alpha + A; \\ \frac{A-B}{A-B+(1+2\alpha)(1-B)}, & (1+3\alpha)B \geq 3\alpha + A. \end{cases}$$

The result is sharp when $(1+3\alpha)B > 3\alpha + A$.

(3) $\sum_{n=2}^{\infty} n^2 a_n \leq \gamma$ *where*

$$\gamma = \begin{cases} \frac{A-B}{(1-B)\alpha}, & (A+1) > 2(\alpha + B - \alpha B); \\ \frac{4(A-B)}{A-B+(1+2\alpha)(1-B)}, & (A+1) \leq 2(\alpha + B - \alpha B). \end{cases}$$

The result is sharp when $(A+1) < 2(\alpha + (1-\alpha)B)$.

(4) $\sum_{n=2}^{\infty} n a_n \leq 2(A-B)/(A-B+(1+2\alpha)(1-B))$. *The result is sharp.*

The functions f represented in the form:

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \mu \in \mathbb{C}. \tag{4}$$

were studied in detail in [7]. Motivated by this, we determine the necessary and sufficient conditions for the functions given by (4) to be in the class $\mathcal{S}^*[A,B]$. The following theorems provide sufficient coefficient inequalities for the normalised analytic functions f with the representation (4) to be in the class $\mathcal{S}^*[A,B]$.

Theorem 1.9. *Let $0 \leq B < A \leq 1$ and $\mu \geq -B/(A-B)$. If the function $f \in \mathcal{A}$ has the representation of the form (4) and b_n satisfies any one of the coefficient inequalities:*

$$(1) \sum_{n=1}^{\infty} n|b_n| \leq \frac{(A-B)\mu}{((1+B)+(A-B)\mu)},$$

$$(2) \sum_{n=2}^{\infty} (n-1)|b_n| \leq \frac{(A-B)\mu - ((1+B)+(A-B)\mu)|b_1|}{2(1+B)+(A-B)\mu},$$

then $f \in \mathcal{S}^*[A,B]$.

Since $n \leq n^2$ for $n \geq 1$, the second part of Theorem 1.9 shows that the inequality

$$\sum_{n=1}^{\infty} n^2|b_n| \leq (A-B)\mu/((1+B)+(A-B)\mu)$$

is sufficient for the function f to belong to $\mathcal{S}^*[A,B]$. Also, for $n \geq 2$, the inequality $2(n-1) \leq n(n-1)$ holds and as a result, the sufficient condition for the function $f \in \mathcal{S}^*[A,B]$, is

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq \frac{2((A-B)\mu - ((1+B)+(A-B)\mu)|b_1|)}{(2(1+B)+(A-B)\mu)},$$

provided $B > 0$ and $\mu \geq -B/(A-B)$.

Theorem 1.10. *Let $-1 \leq B < A \leq 1$, $B < 0$ and $\mu \leq -B/(A-B)$. If the function $f \in \mathcal{A}$ has the form* (4) *and satisfies any one of the coefficient inequalities*

$$(1) \sum_{n=2}^{\infty}(n-1)|b_n| \leq \frac{(A-B)\mu - ((1-B)-(A-B)\mu|b_1|)}{2(1-B)},$$

$$(2) \sum_{n=1}^{\infty} n|b_n| \leq \frac{(A-B)\mu}{(1-B)},$$

then $f \in \mathcal{S}^*[A,B]$.

The inequalities $2(n-1) \leq n(n-1)$ $(n \geq 2)$ and $n \leq n^2$ $(n \geq 1)$ hold. Thus, for $-1 \leq B < A \leq 1$, the inequalities

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq \frac{(A-B)\mu - ((1-B)-(A-B)\mu)|b_1|}{(1-B)}$$

and

$$\sum_{n=1}^{\infty} n^2|b_n| \leq \frac{(A-B)\mu}{(1-B)}$$

are sufficient for $f \in \mathcal{S}^*[A,B]$, provided $B < 0$ and $\mu \leq -B/(A-B)$. A necessary condition for the functions of the form (4) to be in the class $S^*[A,B]$ is given in:

Theorem 1.11. *If the function $f \in S^*[A,B]$, then the following inequality holds:*

$$\sum_{n=1}^{\infty} n^2|b_n|^2 \leq \frac{(A-B)^2\mu^2}{(1-B^2) - 2B(A-B)\mu - (A-B)^2\mu^2}.$$

The inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq (A-B)^2\mu^2/((1-B^2) - 2B(A-B)\mu - (A-B)^2\mu^2)$$

holds trivially, as a consequence of the above theorem and the fact that $n \leq n^2$ for $n \geq 1$.

2. Proofs

Firstly, we prove the following lemma which provides a necessary and sufficient condition for function f to belong to the class $\mathcal{TR}(A,B,\alpha)$.

Lemma 2.1. *Let* $\alpha \in \mathbb{R}$ *and* $-1 \leq B < A \leq 1$. *Let the function* $f \in \mathcal{T}$ *be of the form* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$. *Then the function* $f \in \mathcal{TR}(A, B, \alpha)$ *if and only if the function* f *satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1)a_n \leq A - B. \tag{5}$$

Proof. Let $f \in \mathcal{R}(A, B, \alpha)$. Then, by the definition of subordination there exists a Schwartz function satisfying $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)}\Big(\alpha\frac{zf''(z)}{f'(z)} + 1\Big) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{6}$$

Solving for the function w, we get

$$\begin{aligned} w(z) &= \frac{zf'(z) + \alpha z^2 f''(z) - f(z)}{Af(z) - Bzf'(z) - B\alpha z^2 f''(z)} \\ &= \frac{\sum_{n=2}^{\infty} a_n(-n - \alpha n(n-1) + 1)z^n}{(A-B)z + \sum_{n=2}^{\infty} a_n(-A + Bn + B\alpha n(n-1))z^n}. \end{aligned}$$

Since $\operatorname{Re} w(z) \leq |w(z)| < 1$, we get

$$\operatorname{Re}\Big\{\frac{\sum_{n=2}^{\infty} a_n(-n - \alpha n(n-1) + 1)z^n}{(A-B)z + \sum_{n=2}^{\infty} a_n(-A + Bn + B\alpha n(n-1))z^n}\Big\} < 1$$

As $a_n \in \mathbb{R}$, for $z = r$, the above inequality becomes

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1)a_n r^n < (A-B)r,$$

Letting $r \to 1^-$, we get

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1)a_n < A - B.$$

Conversely, let the inequality (5) holds. We now have to show that $f \in \mathcal{R}(A, B, \alpha)$. For this, we prove that (6) holds and therefore, it is sufficient to show that there exists a Schwarz function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that

$$\frac{\alpha z^2 f''(z) + zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},$$

or, equivalently, it is enough to show that

$$|\alpha z^2 f''(z) + zf'(z) - f(z)| - |Af(z) - B(\alpha z^2 f''(z) + zf'(z))| \leq 0.$$

Since

$$\begin{aligned} \alpha z^2 f''(z) + zf'(z) - f(z) &= -\alpha\sum_{n=2}^{\infty} n(n-1)a_n z^n - \sum_{n=2}^{\infty} na_n z^n + \sum_{n=2}^{\infty} a_n z^n, \\ &= -\sum_{n=2}^{\infty}(n-1)(1+\alpha n)a_n z^n \end{aligned}$$

we have

$$(7)\qquad |\alpha z^2 f''(z) + zf'(z) - f(z)| \leq \sum_{n=2}^{\infty}(n-1)(1+\alpha n)a_n,$$

and similarly,

(8)

$$|Af(z) - B(\alpha z^2 f''(z) + zf'(z))| \geq (A-B) - \sum_{n=2}^{\infty}(A - nB - n^2 B\alpha + nB\alpha)a_n.$$

Using the inequalities (7), (8) and (5) we get

$$\begin{aligned}&\left|\alpha z^2 f''(z) + zf'(z) - f(z)\right| - \left|Af(z) - B(\alpha z^2 f''(z) + zf'(z))\right| \\ &\leq \sum_{n=2}^{\infty}(n^2\alpha + n - n\alpha - 1 + A - nB - n^2 B\alpha + nB\alpha)a_n - (A-B) \\ &= \sum_{n=2}^{\infty}(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1)a_n - (A-B) \leq 0.\end{aligned}$$

This completes the proof of the lemma. □

If we impose the condition $\alpha = 0$ in the above lemma, we get the following lemma:

Lemma 2.2. [4] *Let $-1 \leq B < A \leq 1$. A function $f \in \mathcal{TS}^*[A,B]$ if and only if it satisfies the following inequality:*

$$(9)\qquad \sum_{n=2}^{\infty}(n(1-B) - (1-A))a_n \leq A - B.$$

and the function $f \in \mathcal{TC}[A,B]$ if and only if it satisfies the inequality

$$(10)\qquad \sum_{n=2}^{\infty} n(n(1-B) - (1-A))a_n \leq A - B.$$

With the help of the preceding lemma, we now prove Theorem 1.3 which gives the sufficient condition for the function f to belong to $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$ respectively.

Proof of Theorem 1.3. (a) Let the function f satisfies (1). It can be easily seen that, for $n \geq 2$, the following inequality holds:

$$(n-1)(1-B) + (A-B) \leq \frac{1+A-2B}{2} n(n-1).$$

Consequently, the hypothesis yields

$$\sum_{n=2}^{\infty}((n-1)(1-B) + (A-B))a_n \leq \frac{1+A-2B}{2}\sum_{n=2}^{\infty} n(n-1)a_n \leq A - B.$$

Therefore, by Lemma 2.2, $f \in \mathcal{TS}^*[A,B]$.

Let us now assume that the function f satisfies (2). Then since, for $n \geq 2$, the following inequality can be easily proved:

$$(n-1)(1-B) + (A-B) \leq (1+A-2B)(n-1)$$

Thus,

$$\sum_{n=2}^{\infty}((n-1)(1-B)+(A-B))a_n \leq (1+A-2B)\sum_{n=2}^{\infty}(n-1)a_n \leq A-B.$$

Thus the result holds as a consequence of Lemma 2.2.

We next suppose that (3) holds. Then, for $n \geq 2$, we have the following inequality:

$$(n-1)(1-B)+(A-B) \leq \frac{(1-2B+A)}{4}n^2.$$

Thus,

$$\sum_{n=2}^{\infty}((n-1)(1-B)+(A-B))a_n \leq \frac{(1-2B+A)}{4}\sum_{n=2}^{\infty}n^2 a_n \leq A-B.$$

Hence, by (9) the function f belongs to the class $\mathcal{TS}^*[A,B]$. Finally, let the function f satisfies (4). Then in order to show that f belongs to the class $\mathcal{TS}^*[A,B]$, we use Lemma 2.2 and the following inequality for $n \geq 2$:

$$(n-1)(1-B)+(A-B) \leq \frac{(2-3B+A)}{2}n.$$

We, therefore, have the desired result by Lemma 2.2 as the function f satisfies

$$\sum_{n=2}^{\infty}((n-1)(1-B)+(A-B))a_n \leq \frac{(2-3B+A)}{2}\sum_{n=2}^{\infty}na_n \leq A-B.$$

The functions $f_0 : \mathbb{D} \to \mathbb{C}$ and $f_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = z - \frac{A-B}{1+A-2B}z^2 \qquad \text{and} \qquad f_1(z) = z - \frac{A-B}{2+A-3B}z^2,$$

satisfy the hypothesis of Lemma 2.2 and thus the functions f_0 and f_1 belong to $\mathcal{TS}^*[A,B]$. The function f_0 shows that the bounds obtained in the first three cases are sharp and the function f_1 shows that the bound in the fourth case is sharp.

(b) For the function f satisfying the inequality (1), use the following inequality

$$(n-1)(1-B)+(A-B) \leq (1+A-2B)(n-1) \qquad (n \geq 2),$$

to get

$$\sum_{n=2}^{\infty}n((n-1)(1-B)+(A-B))a_n \leq (1+A-2B)\sum_{n=2}^{\infty}n(n-1)a_n \leq A-B.$$

Thus by Lemma 2.2, $f \in \mathcal{TC}[A,B]$. The result is sharp for the function f_0 given by

$$f_0(z) = z - \frac{A-B}{2(1+A-2B)}z^2.$$

For proving the second part, we again make use of the Lemma 2.2 and the following inequality:

$$n((n-1)(1-B)+(A-B)) \leq (2-3B+A)n^2 \qquad (n \geq 2).$$

The above inequality immediately yields

$$\sum_{n=2}^{\infty} n((n-1)(1-B)+(A-B))a_n \leq (2-3B+A)\sum_{n=2}^{\infty} n^2 a_n \leq A-B$$

which in virtue of inequality (10) proves the result. The sharpness can be seen for the function $f_0 \in \mathcal{TC}[A,B]$ given by

$$f_0(z) = z - \frac{A-B}{4(2+A-3B)}z^2. \qquad \square$$

When $A = 1-\alpha$ and $B = 0$, clearly the class $\mathcal{TS}^*[A,B]$ reduces to the subclass $\mathcal{TS}^*_\alpha$ of $\mathcal{T}$, and hence the the following are sufficient for $f \in \mathcal{TS}^*_\alpha$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2(1-\alpha)/(2-\alpha)$, $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)/(2-\alpha)$, $\sum_{n=2}^{\infty} na_n \leq 2(1-\alpha)/(3-\alpha)$ and $\sum_{n=2}^{\infty} n^2 a_n \leq 4(1-\alpha)/(2-\alpha)$.

The first two results and the last result obtained here are same as proved in [2, Theorem 2.1,Corollary 2.3,Theorem 2.5,], whereas the third coefficient inequality obtained above is an improvement of the already known coefficient bound in [2, Theorem 2.5].

When $A = \alpha$ and $B = -\alpha$, the class $\mathcal{TS}^*[A,B]$ reduces to the subclass $\mathcal{TS}^*[\alpha]$ of $\mathcal{T}$ of starlike functions, and hence the following are sufficient for $f \in \mathcal{TS}^*[\alpha]$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq 4\alpha/(1+3\alpha)$, $\sum_{n=2}^{\infty}(n-1)a_n \leq 2\alpha/(1+3\alpha)$, $\sum_{n=2}^{\infty} na_n \leq 2\alpha)/(1+2\alpha)$ and $\sum_{n=2}^{\infty} n^2 a_n \leq 8\alpha/(1+3\alpha)$.

When $A = 1-\alpha$ and $B = 0$, clearly the class $\mathcal{TC}[A,B]$ reduces to the class $\mathcal{TC}_\alpha$, where $\mathcal{TC}_\alpha$ is the subclass of $\mathcal{T}$ of functions convex of order α, and hence the following are sufficient for $f \in \mathcal{TC}_\alpha$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq (1-\alpha)/(2-\alpha)$ and $\sum_{n=2}^{\infty} n^2 a_n \leq (1-\alpha)/(3-\alpha)$.

The second coefficient inequality obtained above is an improvement of the already known coefficient inequality as in [2, Theorem 2.5] and the first one is same as obtained in [2, Theorem 2.1].

When $A = \alpha$ and $B = -\alpha$, clearly the class $\mathcal{TC}[A,B]$ reduces to the class $\mathcal{TC}[\alpha]$, where $\mathcal{TC}[\alpha]$ is the subclass of $\mathcal{T}$, and hence the following are sufficient for $f \in \mathcal{TC}[\alpha]$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2\alpha/(1+3\alpha)$ and $\sum_{n=2}^{\infty} n^2 a_n \leq \alpha/(1+2\alpha)$.

Proof of Theorem 1.4. (1) Lemma 2.2 along with the inequality

$$(n-1)(1-B) \leq (n-1)(1-B)+(A-B) \qquad n \geq 2,$$

immediately yields

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \sum_{n=2}^{\infty} \frac{n((n-1)(1-B)+(A-B))}{(1-B)}a_n \leq \frac{A-B}{(1-B)}.$$

(2) Using the following inequality:

$$n(1+A-2B) \leq 2((n-1)(1-B)+(A-B)) \qquad n \geq 2,$$

and Lemma 2.2, we get:

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{2n((n-1)(1-B)+(A-B))}{(1+A-2B)}a_n \leq \frac{2(A-B)}{(1+A-2B)}.$$

The function $f_0 \in \mathcal{TC}[A,B]$ given by

$$f_0(z) = z - \frac{A-B}{2(1+A-2B)} z^2,$$

shows that the results are sharp. This completes the proof of the theorem.

□

When $A = 1-\alpha$ and $B = 0$, the class $\mathcal{TC}[A,B]$ reduces to the class $\mathcal{TC}_\alpha$, and hence the following coefficient inequalities follow if $f \in \mathcal{TC}_\alpha$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq 1-\alpha$, $\sum_{n=2}^{\infty} n^2 a_n \leq 2(1-\alpha)/(2-\alpha)$, $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)/2(2-\alpha)$ and $\sum_{n=2}^{\infty} na_n \leq (1-\alpha)/(2-\alpha)$.

When $A = \alpha$ and $B = -\alpha$, clearly the class $\mathcal{TC}(A,B)$ reduces to the class $\mathcal{TC}[\alpha]$, and hence we get the the following coefficient inequalities if $f \in \mathcal{TC}[\alpha]$: $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2\alpha/(1+\alpha)$, $\sum_{n=2}^{\infty} n^2 a_n \leq 4\alpha/(1+3\alpha)$, $\sum_{n=2}^{\infty}(n-1)a_n \leq \alpha/(1+3\alpha)$ and $\sum_{n=2}^{\infty} na_n \leq 2\alpha/(1+3\alpha)$.

Corollary 2.3. *If $f \in \mathcal{TS}^*[A,B]$, then the following inequalities hold:*

(1) $\sum_{n=2}^{\infty} a_n \leq (A-B)/(1+A-2B)$.
(2) $\sum_{n=2}^{\infty} na_n \leq 2(A-B)/(1+A-2B)$.
(3) $\sum_{n=2}^{\infty}(n-1)a_n \leq (A-B)/(1-B)$.

The bounds obtained in the first two cases are sharp.

Proof. The results follow from Theorem 1.4 and the Alexander relation between the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$. It can be directly proved by using Lemma 2.2 by using the inequalities $(1+A-2B) \leq (n-1)(1-B)+(A-B)$, $(1+A-2B)n \leq 2((n-1)(1-B)+(A-B))$ and $(1-B)(n-1) \leq (n-1)(1-B)+(A-B)$ respectively for $n \geq 2$. The sharpness follows by considering the function $f_0(z) = z-(A-B)/(1+A-2B)z^2 \in \mathcal{TS}^*[A,B]$. □

When $A = 1-\alpha$ and $B = 0$, the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*_\alpha$, and hence the following coefficient inequalities follow if $f \in \mathcal{TS}^*_\alpha$: $\sum_{n=2}^{\infty} a_n \leq (1-\alpha)/(2-\alpha)$, $\sum_{n=2}^{\infty} na_n \leq 2(1-\alpha)/(2-\alpha)$ and $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)$.

When $A = \alpha$ and $B = -\alpha$, clearly the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*[\alpha]$, and hence we get the the following coefficient inequalities if $f \in \mathcal{TS}^*[\alpha]$: $\sum_{n=2}^{\infty} a_n \leq 2\alpha/(1+3\alpha)$, $\sum_{n=2}^{\infty} na_n \leq 4\alpha/(1+3\alpha)$. and $\sum_{n=2}^{\infty}(n-1)a_n \leq 2\alpha/(1+3\alpha)$.

Remark 2.4. *For $A = 1-2\alpha$ and $B = -1$, Theorems 1.3, 1.4 and Corollary 2.3 reduce to* [2, Theorems 2.1,2.5,4.4,4.5].

Proof of Theorem 1.5. (1) In [10, Theorem 2], Silverman and Silvia proved that $\mathcal{S}^*[C,D] \subset \mathcal{S}^*[A,B]$ (or $\mathcal{C}[C,D] \subset \mathcal{C}[A,B]$) if and only if the following inequalities hold:

$$\frac{1-A}{1-B} \leq \frac{1-C}{1-D} \quad \text{and} \quad \frac{1+C}{1+D} \leq \frac{1+A}{1+B}.$$

In particular, when $B = D$, both of the above conditions reduce to $A \geq C$. Consequently, if $C \geq C_0 = (A-B+D(1-A)+2\alpha D(1-B))/((1-B)(1+2\alpha))$, then $\mathcal{TS}^*[C_0,D] \subset \mathcal{TS}^*[C,D]$. Hence, we only need to prove that $f \in \mathcal{TS}^*[C_0, D]$. This is proved by making use the following inequality,

(11)

$$(n-1)(1-B)(1+2\alpha)+(A-B) \leq \alpha(1-B)n^2+(1-\alpha)(1-B)n+A-1 \qquad n \geq 2.$$

Now, using the inequalities (5) and (11), it readily follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}((n-1)(1-D)+(C_0-D))a_n \\
&= \sum_{n=2}^{\infty}\Big((n-1)(1-D)+\frac{(A-B)(1-D)}{(1-B)(1+2\alpha)}\Big)a_n \\
&= \sum_{n=2}^{\infty}(1-D)\times\Big(\frac{(n-1)(1+2\alpha)(1-B)+(A-B)}{(1-B)(1+2\alpha)}\Big)a_n \\
&\leq \sum_{n=2}^{\infty}(1-D)\times\Big(\frac{n^2(1-B)\alpha+n(1-B)(1-\alpha)+A-1}{(1-B)(1+2\alpha)}\Big)a_n \\
&\leq \frac{(1-D)(A-B)}{(1-B)(1+2\alpha)} = C_0 - D
\end{aligned}$$

Thus by Lemma 2.2, $f \in \mathcal{TS}^*[C_0, D]$. The function f_0 given by:

$$f_0(z) = z - \frac{A-B}{4\alpha(1-B)+2(1-\alpha)(1-B)+A-1}z^2,$$

satisfies the hypothesis of Lemma 2.1 and hence f_0 belongs to $\mathcal{TR}(A, B, \alpha)$ shows that the result is sharp.

(2) If $C \geq C_0 = (A-B+D(\alpha-A)+BD(1-\alpha))/\alpha(1-B)$, then $\mathcal{TC}[C_0, D] \subset \mathcal{TC}[C, D]$. Thus, it is enough to show that f belongs to $\mathcal{TC}[C_0, D]$. The following inequality holds for $n \geq 2$:

$$n((n-1)\alpha(1-B)+(A-B)) \leq n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1$$

Now, the above inequality together with (5) shows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}n((n-1)(1-D)+(C_0-D))a_n \\
&= \sum_{n=2}^{\infty}n\Big((n-1)(1-D)+\frac{(A-B)(1-D)}{(1-B)\alpha}\Big)a_n \\
&= \sum_{n=2}^{\infty}(1-D)n\Big(\frac{(n-1)\alpha(1-B)+(A-B)}{(1-B)\alpha}\Big)a_n \\
&\leq \frac{(1-D)}{\alpha(1-B)}\sum_{n=2}^{\infty}(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1)a_n \\
&\leq \frac{(1-D)(A-B)}{(1-B)\alpha} = C_0 - D.
\end{aligned}$$

Thus by making use of Lemma 2.2, we get that the function f belongs to the class $\mathcal{TC}[C_0, D]$.

□

Proof of Theorem 1.6. Since, for $n \geq 2$, the following inequality holds:

$$2(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1) \leq (2\alpha(1-B)-2B+A+1)n(n-1),$$

and using this, we see that

$$\begin{aligned}&\sum_{n=2}^{\infty}(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1)a_n\\ &\leq \frac{1}{2}\sum_{n=2}^{\infty}(2\alpha(1-B)-2B+A+1)n(n-1)a_n \leq A-B.\end{aligned}$$

Thus, by Lemma 2.1, $f \in \mathcal{TR}(A,B,\alpha)$. The function $f_0 \in \mathcal{TR}(A,B,\alpha)$ given by

$$f_0(z) = z - \frac{A-B}{4\alpha(1-B)+2(1-\alpha)(1-B)+A-1}z^2,$$

shows that the result is sharp. □

Proof of Theorem 1.7. For $C \leq C_0$, $\mathcal{TC}[C,D] \subset \mathcal{TC}[C_0,D]$. Thus it is e-nough to show that $\mathcal{TC}[C_0,D] \subseteq \mathcal{TR}(A,B,\alpha)$, where $C_0 = (2A-2B+(1+2\alpha-3A+2B-2\alpha B)D)/(1-A-2\alpha(-1+B))$. For $n \geq 2$, the following inequality holds:

$$\begin{aligned}2(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1 \leq A(3-n)+(n-1)(1+2\alpha)\\ +2B(-1+\alpha-n\alpha)\end{aligned}$$

This yields,

$$\begin{aligned}&\sum_{n=2}^{\infty}(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1)a_n\\ &\leq \sum_{n=2}^{\infty}\frac{A(3-n)+(n-1)(1+2\alpha)+2B(-1+\alpha-n\alpha)}{2}a_n\\ &= \sum_{n=2}^{\infty}\frac{(n-1)(1-D)+(C_0-D)}{2(1-D)} \times (1-A-2\alpha(-1+B))a_n\\ &\leq \frac{(C_0-D)}{2(1-D)} \times (1-A-2\alpha(-1+B))a_n\\ &= \frac{2(1-D)(A-B)}{2(1-D)(1-A-2\alpha(-1+B))} \times (1-A-2\alpha(-1+B))a_n\\ &= A-B\end{aligned}$$

Thus by Lemma 2.1 we get $f \in \mathcal{TR}(A,B,\alpha)$. □

Proof of Theorem 1.8. (1) Since $f \in \mathcal{TR}(A,B,\alpha)$, by Lemma 2.1 we have

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)a_n \leq (A-B). \tag{12}$$

For $n \geq 2$, the following inequality holds:

$$\alpha(1-B)n(n-1) \leq (n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1). \tag{13}$$

Then, equations (12) and (13) readily give

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \sum_{n=2}^{\infty} \frac{(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)}{\alpha(1-B)}a_n \leq \frac{(A-B)}{\alpha(1-B)}.$$

(2) When $(1+3\alpha)B < 3\alpha + A$, then for $n \geq 2$,

$$(1-\alpha)(1-B)(n-1) \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1. \tag{14}$$

Then, inequations (14) and (12) give

$$\sum_{n=2}^{\infty}(n-1)a_n \leq \sum_{n=2}^{\infty} \frac{n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1}{(1-\alpha)(1-B)}a_n \leq \frac{(A-B)}{(1-\alpha)(1-B)}.$$

When $(1+3\alpha)B \geq 3\alpha + A$, then for $n \geq 2$ the following inequality holds,

$$(1+A+2\alpha-2B-2\alpha B)(n-1) \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1. \tag{15}$$

Using (12) and (15), we get

$$\sum_{n=2}^{\infty}(n-1)a_n \leq \sum_{n=2}^{\infty}\left(\frac{n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1}{(1+A+2\alpha-2B-2\alpha B)}\right)a_n \leq \frac{(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.$$

(3) When $(A+1) > 2(\alpha+B-\alpha B)$, then the inequality:

$$\alpha(1-B)n^2 \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1 \qquad n \geq 2,$$

together with the inequation (12) gives

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1}{\alpha(1-B)}a_n \leq \frac{(A-B)}{\alpha(1-B)}.$$

When $(A+1) \leq 2(\alpha+B-\alpha B)$, then for $n \geq 2$ the following inequality holds,

$$(1+A+2\alpha-2B-2\alpha B)n^2 \leq 4(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1). \tag{16}$$

Using (12) and (16), we get

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{4(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)}{(1+A+2\alpha-2B-2\alpha B)}a_n \leq \frac{4(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.$$

(4) For $\alpha > 0$, the inequality

$$(1+A+2\alpha-2B-2\alpha B)n \leq 2(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1),$$

together with (12) shows that

$$\sum_{n=2}^{\infty} na_n \leq \sum_{n=2}^{\infty} \frac{2(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)}{(1+A+2\alpha-2B-2\alpha B)} a_n$$
$$\leq \frac{2(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.$$

Sharpness follows by considering the function $f_0 \in \mathcal{TR}[A,B,\alpha]$ given by

$$f_0(z) = z - \frac{A-B}{4\alpha(1-B)+2(1-\alpha)(1-B)+A-1} z^2.$$

□

Remark 2.5. *Replacing $C = 1-2\alpha$ and $D = -1$, Theorems 1.5-1.8 reduce to the results obtained in* [2, Theorems 3.2, 3.3, 4.8, 4.9] *for the class $\mathcal{TR}(\alpha,\beta)$.*

Proof of theorem 1.9. In order to study the necessary and sufficient conditions for the Janowski starlikeness for functions of the form (4), we need the following lemma:

Lemma 2.6. [7, Theorem 2.1] *Suppose that $f \in \mathcal{A}$ has the representation (4) and the coefficients b_n satisfy the inequality*

$$\sum_{n=1}^{\infty}(n+|(A-B)\mu+Bn|)|b_n| \leq (A-B)\mu, \tag{17}$$

where $-1 \leq B \leq A \leq 1$. Then $f \in \mathcal{S}^*[A,B]$.

However, if $B>0$ and $\mu \geq -B/(A-B)$, then inequality (17) reduces to:

$$\sum_{n=1}^{\infty}((1+B)n+(A-B)\mu)|b_n| \leq (A-B)\mu. \tag{18}$$

And if $B<0$ and $\mu \leq -B/(A-B)$, then equation (17) reduces to:

$$\sum_{n=1}^{\infty}((1-B)n-(A-B)\mu)|b_n| \leq (A-B)\mu. \tag{19}$$

(1) For $n \geq 1$, the following inequality holds:

$$(1+B)n+(A-B)\mu \leq ((1+B)+(A-B)\mu)n \tag{20}$$

Thus using the inequality (20), we see that

$$\sum_{n=1}^{\infty}((1+B)n+(A-B)\mu)|b_n| \leq ((1+B)+(A-B)\mu)n|b_n|$$
$$\leq (A-B)\mu.$$

Hence by Lemma 2.6, $f \in \mathcal{S}^*[A,B]$.

(2) For $n \geq 2$, the following inequality holds

$$((1+B)n+(A-B)\mu) \leq (2(1+B)+(A-B)\mu)(n-1). \tag{21}$$

Using equation (21) we see that

$$\sum_{n=1}^{\infty}((1+B)n+(A-B)\mu)|b_n|$$

$$= ((1+B)+(A-B)\mu)|b_1| + \sum_{n=2}^{\infty}((1+B)n+(A-B)\mu)|b_n|$$
$$\leq ((1+B)+(A-B)\mu)|b_1| + \sum_{n=2}^{\infty}(2(1+B)+(A-B)\mu)(n-1)|b_n|$$
$$\leq ((1+B)+(A-B)\mu)|b_1| + (2(1+B)+(A-B)\mu) \times \left(\frac{(A-B)\mu - ((1+B)+(A-B)\mu)|b_1|}{2(1+B)+(A-B)\mu}\right) \leq (A-B)\mu$$

Thus by using Lemma 2.6, $f \in \mathcal{S}^*[A,B]$.

□

Proof of Theorem 1.10. (1) For proving the first part of the theorem, we observe that the following inequality can be proved easily for $n \geq 2$:

$$(1-B)n - (A-B)\mu \leq 2(1+B)(n-1). \tag{22}$$

Therefore, using equation (22) and (19) we see that

$$\sum_{n=1}^{\infty}((1-B)n-(A-B)\mu)|b_n|$$
$$= ((1-B)-(A-B)\mu)|b_1| + \sum_{n=2}^{\infty}((1-B)n-(A-B)\mu)|b_n|$$
$$\leq ((1-B)-(A-B)\mu)|b_1| + \sum_{n=2}^{\infty}2(1-B)(n-1)|b_n|$$
$$\leq ((1-B)-(A-B)\mu)|b_1| + 2(1-B)\times \left(\frac{(A-B)\mu - ((1-B)-(A-B)\mu|b_1|)}{2(1-B)}\right)$$
$$\leq (A-B)\mu.$$

Thus by using Lemma 2.6 $f \in \mathcal{S}^*[A,B]$.

(2) For the second part, we see that the following inequality holds for $n \geq 1$:

$$(1-B)n - (A-B)\mu \leq (1-B)n. \tag{23}$$

Therefore, using equations (23) and (19) we see that

$$\sum_{n=1}^{\infty}(1-B)n-(A-B)\mu|b_n| \leq \sum_{n=1}^{\infty}(1-B)n|b_n|$$
$$\leq (1-B)\times\left(\frac{(A-B)\mu}{(1-B)}\right)$$
$$\leq (A-B)\mu.$$

Hence, by Lemma 2.6 $f \in \mathcal{S}^*[A,B]$.

□

For $A = 1-2\alpha$ and $B=-1$, then the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*(\alpha)$, thus it can be seen that if any of the inequalities $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq$

$(1-\alpha)\mu-\alpha\mu|b_1|$, $\sum_{n=2}^{\infty}(n-1)|b_n| \leq ((1-\alpha)\mu-\alpha\mu|b_1|)/2$, $\sum_{n=2}^{\infty} n^2|b_n| \leq (1-\alpha)\mu$ or $\sum_{n=2}^{\infty} n|b_n| \leq (1-\alpha)\mu$ holds, then $f \in \mathcal{TS}^*(\alpha)$.

When $A=1-\alpha$ and $B=0$, since the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*_{\alpha}$, thus it can be seen that if any of the inequalities $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq ((1-\alpha)\mu-(1-(1-\alpha)\mu)|b_1|)$, $\sum_{n=2}^{\infty}(n-1)|b_n| \leq ((1-\alpha)\mu-(1-(1-\alpha)\mu)|b_1|)/2$, $\sum_{n=2}^{\infty} n^2|b_n| \leq (1-\alpha)\mu$ or $\sum_{n=2}^{\infty} n|b_n| \leq (1-\alpha)\mu$ holds, then $f \in \mathcal{TS}^*_{\alpha}$.

When $A=\alpha$ and $B=-\alpha$, clearly the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*[\alpha]$, thus it can be seen that if any of the inequalities $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2\alpha\mu-((1+\alpha)-2\alpha\mu)|b_1|)/(1+\alpha)$, $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 2\alpha\mu-((1+\alpha)-2\alpha\mu)|b_1|)/2(1+\alpha)$, $\sum_{n=2}^{\infty} n^2|b_n| \leq 2\alpha\mu/(1+\alpha)$ or $\sum_{n=2}^{\infty} n|b_n| \leq 2\alpha\mu/(1+\alpha)$ holds, then $f \in \mathcal{TS}^*[\alpha]$.

Proof of Theorem 1.11. We prove this theorem using the following lemma:

Lemma 2.7. [7, Theorem 2.4] *Every function $f \in S^*[A,B]$ $(-1 \leq B < A \leq 1)$ which has the form* (4) *with $0<\mu<(1-B)/(A-B)$ satisfies the coefficient inequality*

$$\sum_{n=1}^{\infty}((1-B^2)n^2-2nB(A-B)\mu-(A-B)^2\mu^2)|b_n|^2 \leq \mu^2(A-B)^2.$$

For $n \geq 1$, the following inequality holds

$$((1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2)n^2 \leq (1-B^2)n^2-2nB(A-B)\mu-(A-B)^2\mu^2 \tag{24}$$

Therefore, using (24) and Lemma 2.7

$$\begin{aligned}\sum_{n=1}^{\infty} n^2|b_n|^2 &\leq \sum_{n=1}^{\infty}\frac{(1-B^2)n^2-2nB(A-B)\mu-(A-B)^2\mu^2}{(1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2}|b_n|^2\\ &\leq \frac{(A-B)^2\mu^2}{(1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2},\end{aligned}$$

and hence the result. □

For $A=1-2\alpha$ and $B=-1$, then the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*(\alpha)$, thus if $f \in \mathcal{TS}^*(\alpha)$ then

$$\sum_{n=1}^{\infty} n^2|b_n|^2 \leq \frac{(1-\alpha)\mu}{(1-(1-\alpha)\mu)}.$$

When $A=1-\alpha$ and $B=0$, since the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*_{\alpha}$, thus it can be seen that if $f \in \mathcal{TS}^*_{\alpha}$, then

$$\sum_{n=2}^{\infty} n^2|b_n|^2 \leq \frac{(1-\alpha)^2\mu^2}{1-(1-\alpha)^2\mu^2}.$$

When $A=\alpha$ and $B=-\alpha$, clearly the class $\mathcal{TS}^*(A,B)$ reduces to the class $\mathcal{TS}^*[\alpha]$, thus if $f \in \mathcal{TS}^*[\alpha]$, then

$$\sum_{n=2}^{\infty} n^2|b_n|^2 \leq \frac{4\alpha^2\mu^2}{1-\alpha^2+4\alpha^2\mu-4\alpha^2\mu^2}.$$

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